South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 3 (2024), pp. 265-278

DOI: 10.56827/SEAJMMS.2024.2003.20 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

COMPOSITION OF SPECIAL FUNCTIONS ON CERTAIN INTEGRAL FORMULAS

Shristi Mishra and Harish Nagar

Department of Mathematics,
University Institute of Sciences,
Chandigarh University, Mohali, Punjab, INDIA

E-mail : mshristi69@gmail.com, drharishngr@gmail.com

(**Received:** *Sep. 11, 2024* **Accepted:** *Dec. 28, 2024* **Published:** *Dec. 30, 2024)*

Abstract: In this study, we will explore six novel generalized integral formulas that incorporate the combination of k-Struve and Mittag-Leffler functions. We will derive these expressions in the hypergeometric function form. Additionally, we will address specific cases by employing appropriate substitutions. These results are very promising and adaptable, with wide applications in the field of applied science, engineering, and technological problem solving.

Keywords and Phrases: Mac Robert integral, Oberhettinger integral, Lavoie-Trottier integral, K-Struve function, Mittag-Leffler function.

2020 Mathematics Subject Classification: 42A38, 42B10,46F12.

1. Introduction

In the realm of science and technology, integral formulas prove highly valuable for solving pertinent problems. It's important to note that numerous integral methods have been established; however, practical constraints, such as time limitations, can impact their application. Numerous authors, including Brychkov [5], Choi et al. [7], Agarwal et al. [1], Choi and Agarwal [8], Manaria et al. [16], Khan, Kashmin [13], and Nisar et al. [19], have worked on creating a diverse range of special functions that play a crucial role in a multitude of integral formulas and their specific instances [9, 25, 27].

In the field of applied sciences, significant functions are typically defined through improper integrals, series, or finite products. These crucial functions are commonly referred to as special functions. Special functions are a class of mathematical functions that arise in various branches of science, engineering, and mathematics. These functions encompass a wide range of mathematical functions that arise when solving theoretical and practical problems across different branches of mathematics [20] [4]. Special Functions originate from the solution of specific partial differential equations. In mathematics, special functions are defined on natural or complex numbers and can be represented using both integral and series representations [9]. Special functions are very important to research since they are used in the physical, biological, and engineering sciences to solve a variety of problems. Recently, the domains of engineering and physics have made substantial use of two very important functions: the Mittag-Leffler function and the K-Struve function. Our objective here is to introduce several comprehensive integral formulas that incorporate the combination of these two special functions (K-Struve and Mittag-Leffler functions).

1.1. Mittag-Leffler Function

Gösta Mittag-Leffler the Swedish mathematician introduced the termed i.e., Mittag Leffler Function, and is defined as [2] [26],

$$E_\alpha(g) = \sum_{h=0}^{\infty} \frac{(g)^h}{\Gamma(\alpha h + 1)}, \qquad (g \in C; R(\alpha) > 0) \tag{1}$$

where Γ is a gamma function, after this Wiman generalized [24] the Mittag-Leffler function as follows,

$$E_{\alpha,\beta}(g) = \sum_{h=0}^{\infty}, \qquad (g \in C, min(R(\alpha)R(\beta)) > 0)$$

There are number of ways in which Mittag-Leffler function E_α and the extended Mittag-Leffler function $E_{\alpha,\beta}$ can be extended and used in various research area [10]. Prabhakar again introduced the another extension of this function $E_{\alpha,\beta}$ was introduced by Prabhakar Kumar and is defined as,

$$E^{\gamma}_{\alpha,\beta}(g) = \sum_{h=0}^{\infty} \frac{(\gamma)_h}{\Gamma(\alpha h + \beta)} \frac{(g)^h}{h!} \qquad (g \in C, min(R(\alpha)R(\beta)R(\gamma)) > 0) \tag{2}$$

Again, Shukla and Prajapati defined the new extension of this function i.e.,

$$E^{\gamma,r}_{\alpha,\beta}(g) = \sum_{h=0}^{\infty} \frac{(\gamma)_{rh}}{\Gamma(\alpha h+\beta)} \frac{(g)^h}{h!} \qquad (g \in C, min(R(\alpha)R(\beta)R(\gamma)) > 0, r \in (0,1) \bigcup N) \tag{3}$$

Then after Salim and Faraj has also given a new extension of this function and Ozarslan and Yilmaz presented this following new extension,

$$E^{\gamma;d}_{\alpha,\beta}(g;r) = \sum_{h=0}^{\infty} \frac{B_r(\gamma+h, d-\gamma)}{B(\gamma, d-\gamma)} \frac{(d)_h}{\Gamma(\alpha h+\beta)} \frac{(g)^h}{h!}, \tag{4}$$

$$(g \in C, min(R(\alpha)R(\beta)) > 0, R(d) > R(\gamma) > 0, r \in R_0^+)$$

So, by considering Eq. 3 and Eq. 4 we have concluded by defining a new extension of this function,

$$E^{\gamma,d;b}_{\zeta,\alpha,\eta}(g;r) = \sum_{h=0}^{\infty} \frac{B_r(\gamma+hb, d-\gamma)}{B(\gamma, d-\gamma)} \frac{(d)_{h,b}}{\Gamma(\zeta h+\alpha)} \frac{(g)^h}{(\eta)_{ht} h!} \tag{5}$$

$$(b \in R^+; min(R(\zeta), R(\alpha, R(\eta) > 0; R(d) > R(\gamma) > 0; r \in R_{0^+}))$$

Where $B_r(s,t)$ is the extended beta function,

$$B_r(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} e^{\frac{-r}{t(1-t)}} dt \qquad (r \in R_0^+; min(R(u), R(v)) > 0)$$

If $r = 0$, it reduces to the particular case of well-known beta function

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt \qquad min(R(u), R(v)) > 0$$

$$= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \qquad u, v \in C/Z_0^-$$

And $(d)_{h,b} = \frac{\Gamma(d+hb)}{\Gamma(d)}$, is the extended pochammer symbol [16].

1.2. K-Struve Function

Hermann Struve in 1882 introduced the K-Struve function. Recently, M. F. Nisar et al. studied various properties of Struve function and introduced k-Struve function $S^k_{\varsigma,z}$ is defined by [12] [21],

$$S^k_{\varsigma,s}(d) = \sum_{g=0}^{\infty} \frac{(-s)^g}{\Gamma_k\left(gk+\varsigma+\frac{3k}{2}\right)\Gamma\left(g+\frac{3}{2}\right) g!} \left(\frac{d}{2}\right)^{2g+\frac{\varsigma}{k}+1} \tag{6}$$

After this the generalized Wright Hypergeometric Function ${}_a\psi_b(w)$ is given by the series [23, 28, 29],

$$ {}_a\psi_b(w) =_a \psi_b \left[\begin{matrix}(p_i,\gamma_j)_{1,a}\\ (q_{i,v_j})_{1,b}\end{matrix}\right]; w = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{a}\Gamma(p_i+\gamma_j k)}{\prod_{j=0}^{b}\Gamma(q_j+v_j k)}\frac{w^n}{n!} $$

Where $p_i, q_j \in C$, and real $\gamma_i, v_j \in R(i = 1,2,3,4,5,...,a; j = 1,2,3,4,5,...,b)$. For the different values of the argument $c \in C$, this function behaves asymptotically. In the work of E.M. Wright, it has been found that,

$$ \sum_{j=1}^{b} v_j - \sum_{i=1}^{a}\gamma_i > -1 $$

While defining the properties of generalized Wright function it was investigated that ${}_a\psi_b(c), c \in C$ is an entire function under some condition. Sharma and Devi [23] introduced and investigated the following extended Wright generalized hypergeometric function,

$$ {}_{a+1}\psi_{b+1}\left[\begin{matrix}(p_i,\gamma_j)_{1,a}, & (\gamma,1)\\ (q_i,v_j)_{1,b}, & (c,1)\end{matrix}\right] w; p = \frac{1}{c-\gamma}\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{a}\Gamma(p_i+\gamma_j n)}{\prod_{j=0}^{b}\Gamma(q_j+v_j n)}\frac{B_p(\gamma+n, c-\gamma)w^n}{n!} \tag{7} $$

Where $R(p) > 0, R(c) > R(\gamma) > 0, a, b \in N_0; p_i, q_j \in C$ and real $\gamma_i, v_j \in R(i = 1,2,3,4,5,...,a; j = 1,2,3,4,5,...,b)$. The empty product is understood to be 1 and the summation is assumed to be convergent.

1.3. Integral Formulas

For our present investigation we need some integral formulas given by Mac Robert [15], Oberhettinger [22] and Lavoie-Trottier [14] in equation respectively are as follows, see also [11] [17]

$$ \int_0^1 n^{\mu-1}(1-n)^{\xi-1}[wn+h(1-n)]^{-\mu-\xi}dn = \frac{1}{w^\mu h^\xi}\frac{\Gamma(\mu)\Gamma(\xi)}{\Gamma(\mu+\xi)} \tag{8} $$

Provided that $R(\mu) > 0, R(\xi) > 0, w$ and h are nonzero constants so the expressions $(wn + h(1-n))$, where $0 \le n \le 1$,

$$ \int_0^\infty p^{\xi-1}(p+w+\sqrt{(p^2+2wp)})^{-\mu}dp = 2\mu w^{-\mu}\frac{\Gamma(2\xi)\Gamma(\mu-\xi)}{\Gamma(1+\mu+\xi)} \tag{9} $$

Provided that $0 < R(\xi) < R(\mu)$

$$ \int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1}dt = \left(\frac{2}{3}\right)^{2\mu}\frac{\Gamma(\mu)\Gamma(\xi)}{\Gamma(\mu+\xi)} \tag{10} $$

Provided that $R(\mu) > 0, R(\xi) > 0$.

2. Integral Formulas Involving Composition of Special Functions

The generalised integral formulas that we create in this section are stated in terms of generalised hypergeometric functions through the use of the product of the Mittag-Leffler and K-Struve functions, studied by Chaudhary et al. [6] and Nagar et al. [18].

Theorem A. *The following integral holds true for* $\mu, \xi \in C$ *with* $R(\mu) > 0, R(\xi) > 0, R(\varsigma) > 0, R(\zeta), R(\eta), R(\alpha) > 0,\ R(d) > R(\gamma 0) > 0, r \in R_0^+, b \in R^+\ t > 0$ *we have*

$$\int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1} E^{\zeta,\alpha,\eta}_{\gamma,d,b}; S^k_{\varsigma,s}\left[n\left(1-\frac{t}{4}\right)(1-t)^2\right]dt$$
$$= \left(\frac{2}{3}\right)^{2\mu}\left(\frac{n}{2}\right)^{\frac{\varsigma}{k}+1}\frac{1}{k^{\frac{\varsigma}{k}+\frac{1}{2}}}\frac{\Gamma(\eta)\Gamma(\mu)}{\Gamma(\gamma)\Gamma(d-\gamma)}$$
$$\times {}_3\psi_6\left[\begin{matrix} & & (d,b) & (\frac{\varsigma}{k}+\xi+1,3) & (\gamma,1) & \\ (\alpha,\zeta) & (\eta,t) & (\frac{3}{2},1) & (\frac{\varsigma}{k}+\frac{1}{2},1) & (\mu+\frac{\varsigma}{k}+\xi+1,3) & (d,1)\end{matrix}\right.$$
$$\left.;\left(\left(\frac{-sn^3}{4k}\right)^m; r\right)\right.$$

Proof. In order to establish main result as in theorem 2.1, we denote the left-hand side by I_1 and then by using equation 5 and 6, we have

$$I_1 = \int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1}\sum_{m=0}^{\infty}\frac{B_r(\gamma+mb, d-\gamma)(d)_{mb}(-s)^m}{B(\gamma, d-\gamma)\Gamma(\varsigma m+\alpha)(\eta)_{mt} m!}$$
$$\times\frac{1}{\Gamma_k\left(mk+\varsigma+\frac{3k}{2}\right)2^{2m+\frac{\varsigma}{k}+1}}\times d^{3m+\frac{\varsigma}{k}+1}dt$$

Now by adjusting the order of integration and summation,

$$I_1 = \sum_{m=0}^{\infty}\frac{B_r(\gamma+mb, d-\gamma)(d)_{mb}(-s)^m}{B(\gamma, d-\gamma)\Gamma(\zeta m+\alpha)(\eta)_{mt} m!\Gamma_k\left(mk+\varsigma+\frac{3}{2}\right)\Gamma\left(m+\frac{3}{2}\right)2^{2m+\frac{\varsigma}{k}+1}}\times n^{3m+\frac{\varsigma}{k}+1}$$
$$\times t^{\mu-1}(1-t)^{2\left(3m+\frac{\varsigma}{k}+1+\xi\right)-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\left(3m+\frac{\varsigma}{k}+1+\xi\right)-1}dt$$

Now by making the use of equation 10, and further simplification and rearranging the terms we get

$$I_1 = \sum_{m=0}^{\infty}\frac{B_r(\gamma+mb, d-\gamma)(d)_{mb}(-s)^m}{B(\gamma, d-\gamma)\Gamma(\zeta m+\alpha)(\eta)_{mt} m!\Gamma_k\left(mk+\varsigma+\frac{3}{2}\right)\Gamma\left(m+\frac{3}{2}\right)2^{2m+\frac{\varsigma}{k}+1}}\times n^{3m+\frac{\varsigma}{k}+1}$$
$$\times\left(\frac{2}{3}\right)^{2\mu}\frac{\Gamma(\mu)\Gamma\left(3m+\frac{\varsigma}{k}+1+\xi\right)}{\Gamma(\mu+3m+\frac{\varsigma}{k}+1+\xi)}$$

Furthermore, by simplifying the preceding equation and employing the generalized Wright hypergeometric function, and utilizing equation 7, we achieve the intended outcome.

Theorem B. *The following integral holds true for* $\mu, \xi \in C$ *with* $R(\mu) > 0, R(\xi) > 0, R(\varsigma) > 0$, $R(\zeta), R(\eta), R(\alpha) > 0$, $R(d) > R(\gamma 0) > 0, r \in R_0^+, b \in R^+$ $t > 0$ *we have*

$$\int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1} E_{\gamma,d,b}^{\zeta,\alpha,\eta}; S_{\varsigma,s}^{k}\left[nt\left(1-\frac{t}{4}\right)^2\right] dt$$
$$= \left(\frac{2}{3}\right)^{2\left(\frac{\varsigma}{k}+\mu+1\right)}\left(\frac{n}{2}\right)^{\frac{\varsigma}{k}+1}\frac{1}{k^{\frac{\varsigma}{k}+\frac{1}{2}}}\frac{\Gamma(\eta)\Gamma(\xi)}{\Gamma(\gamma)\Gamma(d-\gamma)}$$
$$\times {}_3\psi_6\left[\begin{matrix} (d,b) & (\frac{\varsigma}{k}+\xi+1,3) & (\gamma,1) \\ (\alpha,\zeta) \quad (\eta,t) \quad (\frac{3}{2},1) & & (\frac{\varsigma}{k}+\frac{3}{2},1) \\ (\mu+\frac{\varsigma}{k}+\xi+1,3) & (d,1) & \end{matrix}\right];\left(\left(\frac{s2^4n^3}{3^6k}\right)^m;r\right)$$

Proof. In order to establish main result as in theorem 2.2, we denote the left-hand side by I_2 and then by using definition of k-Struve and extended Mittag-leffler function, then we have,

$$I_2 = \int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1}\sum_{m=0}^{\infty}\frac{B_r(\gamma+mb,d-\gamma)(d)_{mb}(-s)^m}{B(\gamma,d-\gamma)\Gamma(\zeta+\alpha)(\eta)_{mt}m!}$$
$$\times\frac{1}{\Gamma_k\left(mk+\varsigma+\frac{3k}{2}\right)\Gamma\left(m+\frac{3}{2}\right)2^{2m+\frac{\varsigma}{k}+1}}\times d^{3m+\frac{\varsigma}{k}+1}dt$$

Now by adjusting the order of integration and summation,

$$I_2 = \sum_{m=0}^{\infty}\frac{B_r(\gamma+mb,d-\gamma)(d)_{mb}(-s)^m}{B(\gamma,d-\gamma)\Gamma(\zeta m+\alpha)(\eta)_{mt}m!\Gamma_k\left(mk+\varsigma+\frac{3k}{2}\right)\Gamma\left(m+\frac{3}{2}\right)2^{2m+\frac{\varsigma}{k}+1}}\times n^{3m+\frac{\varsigma}{k}+1}$$
$$\int_0^1 t^{3m+\frac{\varsigma}{k}+1+\mu}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\left(3m+\frac{\varsigma}{k}+1+\mu\right)-1}\left(1-\frac{t}{4}\right)^{\xi-1}dt$$

Now by making the use of equation 10, and further simplification and rearranging the terms we get,

$$I_2 = \sum_{m=0}^{\infty}\frac{B_r(\gamma+mb,d-\gamma)(d)_{mb}(-s)^m}{B(\gamma,d-\gamma)\Gamma(\zeta m+\alpha)(\eta)_{mt}m!\Gamma_k\left(mk+\varsigma+\frac{3k}{2}\right)\Gamma\left(m+\frac{3}{2}\right)2^{2m+\frac{\varsigma}{k}+1}}\times n^{3m+\frac{\varsigma}{k}+1}$$
$$\times\left(\frac{2}{3}\right)^{2\left(3m+\frac{\varsigma}{k}+1+\mu\right)}\frac{\Gamma(\xi)\Gamma\left(3m+\frac{\varsigma}{k}+1+\mu\right)}{\Gamma\left(\mu+3m+\frac{\varsigma}{k}+1+\xi\right)}$$

Furthermore, by simplifying the preceding equation and employing the generalized Wright hypergeometric function, and utilizing equation 7, we achieve the intended outcome.

Theorem C. *The following integral holds true for $\mu, \xi \in C$ with $R(\mu) > 0$,$R(\xi) > 0$,$R(\varsigma) > 0$, $R(\zeta), R(\eta), R(\alpha) > 0$, $R(d) > R(\gamma 0) > 0$,$r \in R_0^+, b \in R^+$ $t > 0$ we have*

$$\int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1} E^{\gamma,d,b}_{\zeta,\alpha,\eta}; S^k_{\varsigma,s}\left[nt(1-t)^2\left(1-\frac{t}{3}\right)^2\left(1-\frac{t}{4}\right)\right]dt$$

$$= \left(\frac{2}{3}\right)^{2\left(\frac{\varsigma}{k}+\mu+1\right)}\left(\frac{n}{2}\right)^{\frac{\varsigma}{k}+1}\frac{1}{k^{\frac{\varsigma}{k}+\frac{1}{2}}}\frac{\Gamma(\eta)}{\Gamma(\gamma)\Gamma(d-\gamma)}$$

$$\times {}_4\psi_6\left[\begin{matrix}(d,b) & \left(\frac{\varsigma}{k}+\mu+1,3\right) & \left(\frac{\varsigma}{k}+\xi+1,3\right) & (\gamma,1)\\ (\alpha,\zeta) & (\eta,t) & \left(\frac{3}{2},1\right) & \left(\frac{\varsigma}{k}+\frac{3}{2},1\right)\\ \left(\mu+\frac{2\varsigma}{k}+\xi+2,6\right) & (d,1) & & \end{matrix}\right];\left(\left(\frac{-s2^4n^3}{3^6k}\right)^m;r\right)$$

Proof. Similar as the above theorems.

Theorem D. *Let us suppose that $R(\mu) > 0$, $R(\xi) > 0$ and $R\varsigma) > 0, R(\zeta), R(\alpha)$, $R(\eta) > 0$ and $R(d) > R(\gamma) > 0; r \in R_0^+, w$ and h are nonzero constants and $0 \le n \le 1$ then*

$$\int_0^1 n^{\mu-1}(1-n)^{\xi-1}[wn+h(1-n)]^{-\mu-\xi}E^{\gamma,d,b}_{\zeta,\alpha,\eta}S^k_{\varsigma,s}\left(\frac{2whn(1-n)}{(wn+h(-n))^2},\zeta,\eta;r\right)dn$$

$$= \frac{\Gamma(\eta)w^{-\mu}h^{-\xi}}{\Gamma(\gamma)\Gamma(d-\gamma)k^{\frac{\varsigma}{k}+\frac{1}{2}}}$$

$${}_4\psi_6\left[\begin{matrix}(d,b) & \left(\frac{\varsigma}{k}+\mu+1,3\right) & \left(\frac{\varsigma}{k}+\xi+1,3\right) & (\gamma,1)\\ (\alpha,\zeta) & (\eta,t) & \left(\frac{3}{2},1\right) & \left(\frac{\varsigma}{k}+\frac{3}{2},1\right)\\ \left(\mu+\frac{2\varsigma}{k}+\xi+2,6\right) & (d,1) & & \end{matrix}\right];\left(\left(\frac{-2s}{w^3h^3k}\right)^m;r\right)$$

Proof. Similar as the above theorems.

Theorem E. *Let us suppose that $0 < R(\xi) < R(\mu), w \in N, R(\alpha)$, $R(\eta) > 0$ $R(d) > R(\gamma) > 0, r \in R_0^+$, then*

$$\int_0^\infty p^{\xi-1}\left(p+w+\sqrt{(p^2+2wp)}\right)^{-\mu}E^{\gamma,d,b}_{\zeta,\alpha,\eta}S^k_{\varsigma,s}\left(\frac{n}{p+w+\sqrt{(p^2+2wp)}},\zeta,\eta;r\right)dp$$

$$= \frac{\Gamma(2\xi)\Gamma(\eta)2\left(\mu+\frac{\varsigma}{k}+1\right)n^{\frac{\varsigma}{k}+1}w^{-\left(\mu+\frac{\varsigma}{k}-\xi+1\right)}}{\Gamma(\gamma)\Gamma(d-\gamma)k^{\frac{\varsigma}{k}+\frac{1}{2}}2^{\frac{\varsigma}{k}+\xi+1}}$$

$$\times {}_3\psi_6\left[\begin{matrix}(d,b) & \left(\frac{\varsigma}{k}+\mu-\xi+1,3\right) & (\gamma,1)\\ (\alpha,\zeta) & (\eta,t) & \left(\frac{3}{2},1\right) & \left(\frac{\varsigma}{k}+\frac{3}{2},1\right)\\ \left(\mu+\frac{\varsigma}{k}+\xi+2,3\right) & (d,1) & \end{matrix}\right];\left(6m\left(\frac{-sn^3w^{-3}}{4k}\right)^m;r\right)$$

Proof: Similar as above.

Theorem F. *Let us suppose that* $0 < R(\xi) < R(\mu), w \in N, R(\alpha), R(\eta) > 0$ $R(d) > R(\gamma) > 0, r \in R_0^+$, *then*

$$\int_0^\infty p^{\xi-1}\left(p+w+\sqrt{(p^2+2wp)}\right)^{-\mu} E^{\gamma,d,b}_{\zeta,\alpha,\eta} S^k_{\varsigma,s}\left(\frac{np}{p+w+\sqrt{(p^2+2wp)}},\zeta,\eta;r\right)dp$$

$$= \frac{\Gamma(\eta)\Gamma(\mu-\xi)2\left(\mu+\frac{\varsigma}{k}+1\right)n^{\frac{\varsigma}{k}+1}w^{(\xi-\mu)}}{\Gamma(\gamma)\Gamma(d-\gamma)k^{\frac{\varsigma}{k}+\frac{1}{2}}2^{\frac{2\varsigma}{k}+\xi+2}}$$

$$\times {}_3\psi_6\left[\begin{matrix} & & (d,b) & \left(\frac{2\varsigma}{k}+2\xi+2,6\right) & (\gamma,1) & \\ (\alpha,\zeta) & (\eta,t) & \left(\frac{3}{2},1\right) & \left(\frac{\varsigma}{k}+\frac{3}{2},1\right) & \left(\mu+\frac{2\varsigma}{k}+\xi+3,6\right) & (d,1)\end{matrix}\right]$$

$$\left(6m\left(\frac{-sn^3}{2^5k}\right)^m;r\right)$$

Proof: Similar as above.

3. Special Case

In this section we shall mention some of the very interesting results in the form of many corollaries. We are going to find new integral formulas by substituting particular values.

Corollary A. *If in the result of theorem* 2.1 *we put* $\mu=\rho+j$ *and* $\xi=\rho$ *and after doing some algebra we get the following result*

$$\int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1} E^{\zeta,\alpha,\eta}_{\gamma,d,b}; S^k_{\varsigma,s}\left[n\left(1-\frac{t}{4}\right)(1-t)^2\right]dt$$

$$= \left(\frac{2}{3}\right)^{2(\rho+j)}\left(\frac{n}{2}\right)^{\frac{\varsigma}{k}+1}\frac{1}{k^{\frac{\varsigma}{k}+\frac{1}{2}}}\frac{\Gamma(\eta)\Gamma(\rho+j)}{\Gamma(\gamma)\Gamma(d-\gamma)}$$

$$\times {}_3\psi_6\left[\begin{matrix} (d,b) & \left(\frac{\varsigma}{k}+\xi+1,3\right) & (\gamma,1) \\ (\alpha,\zeta) \quad (\eta,t) & \left(\frac{3}{2},1\right) & \left(\frac{\varsigma}{k}+\frac{1}{2},1\right) \\ \left(\rho+j+\frac{\varsigma}{k}+\xi+1,3\right) & & (d,1)\end{matrix}\right];\left(\left(\frac{-sn^3}{4k}\right)^m;r\right)$$

Corollary B. *If in the result of theorem* 2.1 *we put* $k=\eta=\gamma=b=n=\alpha=t=r=\zeta=1$ *and after doing certain algebra we get the following result which is very much close to the integral formula on Struve Function* [3].

$$\int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1} E^{\zeta,\alpha,\eta}_{\gamma,d,b}; S^k_{\varsigma,s}\left[n\left(1-\frac{t}{4}\right)(1-t)^2\right]dt$$
$$= \left(\frac{2}{3}\right)^{2\mu}\left(\frac{1}{2}\right)^{\varsigma+1}\Gamma(\mu)$$
$$\times {}_3\psi_4\left[\begin{matrix} & (d,1) & (\varsigma+\xi+1,3) & (1,1) \\ (1,1) & (\frac{3}{2},1) & (\varsigma+\frac{3}{2},1) & (\mu+\varsigma+\xi+1,3)\end{matrix}\right];\left(\left(\frac{-s}{4}\right)^m\right)$$

Corollary C. *If in theorem 2.2 we put $\mu=\rho$ and $\xi=\rho+j$, after doing some algebra we get the following result*

$$\int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1} E^{\zeta,\alpha,\eta}_{\gamma,d,b}; S^k_{\varsigma,s}\left[nt\left(1-\frac{t}{4}\right)^2\right]dt$$
$$= \left(\frac{2}{3}\right)^{2\left(\frac{\varsigma}{k}+\rho+1\right)}\left(\frac{n}{2}\right)^{\frac{\varsigma}{k}+1}\frac{1}{k^{\frac{\varsigma}{k}+\frac{1}{2}}}\frac{\Gamma(\eta)\Gamma(\rho+j)}{\Gamma(\gamma)\Gamma(d-\gamma)}$$
$$\times {}_3\psi_6\left[\begin{matrix} (d,b) & (\frac{\varsigma}{k}+\rho+j+1,3) & (\gamma,1) \\ (\alpha,\zeta) & (\eta,t) & (\frac{3}{2},1) & (\frac{\varsigma}{k}+\frac{3}{2},1) \\ & (\mu+\frac{\varsigma}{k}+\rho+j+1,3) & (d,1)\end{matrix}\right];\left(\left(\frac{s2^4n^3}{3^6k}\right)^m;r\right)$$

Corollary D. *If in theorem 2.3 we put $k=\eta=\gamma=b=n=\alpha=t=r=\zeta=1$ and after doing certain algebra we get the following result*

$$\int_0^1 t^{\mu-1}(1-t)^{2\xi-1}\left(1-\frac{t}{3}\right)^{2\mu-1}\left(1-\frac{t}{4}\right)^{\xi-1} E^{\gamma,d,b}_{\zeta,\alpha,\eta}; S^k_{\varsigma,s}$$
$$\left[nt(1-t)^2\left(1-\frac{t}{3}\right)^2\left(1-\frac{t}{4}\right)\right]dt$$
$$= \left(\frac{2}{3}\right)^{2(\varsigma+\mu+1)}\left(\frac{1}{2}\right)^{\varsigma+1}$$
$$\times {}_4\psi_4\left[\begin{matrix} (d,1) & (\varsigma+\mu+1,3) & (\varsigma+\xi+1,3) & (1,1) \\ (1,1) & (\frac{3}{2},1) & (\varsigma+\frac{3}{2},1) & (\mu+2\varsigma+\xi+2,6)\end{matrix}\right];\left(\left(\frac{-s2^4}{3^6}\right)^m\right)$$

Corollary E. *If in theorem 2.4 we put $k=\eta=\gamma=b=n=\alpha=t=r=\zeta=1$ and after doing certain algebra we get the following result*

$$\int_0^1 n^{\mu-1}(1-n)^{\xi-1}[wn+h(1-n)]^{-\mu-\xi} E^{\gamma,d,b}_{\zeta,\alpha,\eta} S^k_{\varsigma,s}\left(\frac{2whn(1-n)}{(wn+h(-n))^2},\zeta,\eta;r\right)dn$$
$$= w^{-\mu}h^{-\xi}$$
$${}_4\psi_4\left[\begin{matrix} (d,1) & (\varsigma+\mu+1,3) & (\varsigma+\xi+1,3) & (1,1) \\ (1,1) & (\frac{3}{2},1) & (\varsigma+\frac{3}{2},1) & (\mu+2\varsigma+\xi+2,6)\end{matrix}\right];\left(\left(\frac{-2s}{w^3h^3}\right)^m\right)$$

Corollary F. *If in theorem 2.5 we put $\mu = \rho + j$ and $\xi = \rho$, after doing some algebra we get the following result*

$$\int_0^\infty p^{\xi-1}\left(p+w+\sqrt{(p^2+2wp)}\right)^{-\mu} E_{\zeta,\alpha,\eta}^{\gamma,d,b} S_{\varsigma,s}^{k}\left(\frac{n}{p+w+\sqrt{(p^2+2wp)}},\zeta,\eta;r\right)dp$$

$$= \frac{\Gamma(2\rho)\Gamma(\eta)2\left(\rho+j+\frac{\varsigma}{k}+1\right)n^{\frac{\varsigma}{k}+1}w^{-\left(j+\frac{\varsigma}{k}+1\right)}}{\Gamma(\gamma)\Gamma(d-\gamma)k^{\frac{\varsigma}{k}+\frac{1}{2}}2^{\frac{\varsigma}{k}+\rho+1}}$$

$$\times {}_3\psi_6\left[\begin{matrix} (d,b) \quad \left(\frac{\varsigma}{k}+j+1,3\right) \quad (\gamma,1) \\ (\alpha,\zeta) \quad (\eta,t) \quad \left(\frac{3}{2},1\right) \quad \left(\frac{\varsigma}{k}+\frac{3}{2},1\right) \\ \left(2\rho+j+\frac{\varsigma}{k}+2,3\right) \quad (d,1) \end{matrix}\right];\left(6m\left(\frac{-sn^3w^{-3}}{4k}\right)^m;r\right)$$

Corollary G. *If in theorem 2.6 we put $\mu = \rho$ and $\xi = \rho + j$, after doing some algebra we get the following result*

$$\int_0^\infty p^{\xi-1}\left(p+w+\sqrt{(p^2+2wp)}\right)^{-\mu}$$

$$E_{\zeta,\alpha,\eta}^{\gamma,d,b} S_{\varsigma,s}^{k}\left(\frac{np}{p+w+\sqrt{(p^2+2wp)}},\zeta,\eta;r\right)dp$$

$$= \frac{\Gamma(\eta)\Gamma(-j)2\left(\rho+\frac{\varsigma}{k}+1\right)n^{\frac{\varsigma}{k}+1}w^{j}}{\Gamma(\gamma)\Gamma(d-\gamma)k^{\frac{\varsigma}{k}+\frac{1}{2}}2^{\frac{2\varsigma}{k}+\rho+j+2}}$$

$$\times {}_3\psi_6\left[\begin{matrix} (d,b) \quad \left(\frac{2\varsigma}{k}+2(\rho+j)+2,6\right) \quad (\gamma,1) \\ (\alpha,\zeta) \quad (\eta,t) \quad \left(\frac{3}{2},1\right) \quad \left(\frac{\varsigma}{k}+\frac{3}{2},1\right) \quad \left(2\rho+\frac{2\varsigma}{k}+j+3,6\right) \quad (d,1) \end{matrix}\right]$$

$$\left(6m\left(\frac{-sn^3}{2^5k}\right)^m;r\right)$$

4. Conclusion

We have found six new generalised integral formulas in the course of this inquiry. These formulas are built by combining the Mittag-Leffler function and k-Struve function. The results are given as product-form hypergeometric functions. This was accomplished by leveraging the properties of product of two power series. Additionally, we have examined specific scenarios by applying appropriate substitutions. The potential future avenues for these integrals are promising. It is possible to define numerous other remarkable integrals by utilizing different variations of these two functions, trigonometric and hyperbolic functions, along with suitable parametric substitutions, special functions combined with various types of polynomials or multivariable polynomials. These extensions have amazing potential for success. It is noteworthy that the results we have showcased are broad in scope and have real-world implications within the domains of science and technology.

References

[1] Agarwal, P., Jain, S., Agarwal, S. and Nagpal, M., On a new class of integrals involving Bessel functions of the first kind, Commun. Numer. Anal, 2014, (2014).

[2] Andri´c, M., Farid, G. and Peˇcari´c, J., A further extension of Mittag-Leffler function, Fractional Calculus and Applied Analysis, 21(5) (2018), 1377–1395.

[3] Baricz, A. and Pog´any, T. K., Integral representations and summations of the modified Struve function, Acta Mathematica Hungarica, 141 (2013), 254–281.

[4] Belgacem, F., Applications with the Sumudu transform to bessel functions and equations, Appl. Math. Sci, 4(74) (2010), 3665–3686.

[5] Brychkov, Y.A., Handbook of special functions: derivatives, integrals, series and other formulas, CRC press, 2008.

[6] Chaudhary, M. P., Certain Aspects of Special Functions and Integral Operators, LAMBERT Academic Publishing, Saarbrucken, Germany, (2014).

[7] Choi, J., Agarwal, P., Mathur, S. and Purohit, S. D., Certain new integral formulas involving the generalized Bessel functions, Bulletin of the Korean Mathematical Society, 51(4) (2014), 995–1003.

[8] Choi, J. and Agarwal, P., Certain unified integrals involving a product of Bessel functions of first kind, Honam Math. J, 35(4) (2013), 667–677.

[9] Choi, J., Kachhia, K.B., Prajapati, J. C. and Purohit, S. D., Some integral transforms involving extended generalized Gauss hypergeometric functions, Commun. Korean Math. Soc, 31(4) (2016), 779–790.

[10] Haubold, H. J., Mathai, A. M., Saxena et al., R. K., Mittag-Leffler functions and their applications, Journal of applied mathematics, 2011, (2011).

[11] Jangid, N. K., Joshi, S. and Purohit, S. D., Some unified integral formulae associated with Hurwitz-Lerch zeta function, Palestine Journal of Mathematics, 12(1), (2023).

[12] Kabra, S., Nagar, H., Nisar, K. S. and Suthar, D., The Marichev-Saigomaeda fractional calculus operators pertaining to the generalized-Struve function, Applied Mathematics and Nonlinear Sciences, 5(2) (2020), 593–602.

[13] Khan, N. and Kashmin, T., Some integrals for the generalized Besselmaitland functions, Electronic Journal of Mathematical Analysis and Applications, 4(2) (2016), 139–149.

[14] Lavoie, J. and Trottier, G., On the sum of certain Appell's series, Ganita, 20(1) (1969), 31–32.

[15] MacRobert, T., Beta-function formulae and integrals involving functions, Mathematische Annalen, 142(5) (1961), 450–452.

[16] Menaria, N., Nisar, K. and Purohit, S., On a new class of integrals involving product of generalized Bessel function of the first kind and general class of polynomials, Acta Univ. Apulensis, 46 (2016), 97–105.

[17] Nagar, H., Maheshwari, P. and Nisar, K. S., Certain integral associated with the Bessel function, Journal of Science and Arts, 18(3) (2018), 621–630.

[18] Nagar, H., and Mishra, S., Composition of pathway fractional integral operator on product of special functions, Journal of Ramanujan Society of Mathematics and Mathematical Sciences, 10(1), (2022).

[19] Nisar, K., Parmar, R. and Abusufian, A., Certain new unified integrals associated with the generalized k-bessel function, Far East Journal of Mathematical Sciences, 100(9) (2016), 1533.

[20] Nisar, K., Suthar, D., Purohit, S. and Aldhalfallah, M., Some unified integrals associated with the generalized Struve function, in Proceedings of the Jangjeon Mathematical Society, 20(2) (2017), 261–267.

[21] Nisar, K. S., Mondal, S. R. and Choi, J., Certain inequalities involving the k-Struve function, Journal of inequalities and applications, 2017(1) (2017), 1–8.

[22] Oberhettinger, F., Tables of Mellin transforms Springer-Verlag Berlin Heidelberg New York, 1974.

[23] Rao, S. B., Prajapati, J. C., Patel, A. D. and Shukla, A. K., Some properties of Wright-type generalized hypergeometric function via fractional calculus, Advances in difference equations, 2014 (2014), 1–11.

[24] Salim, T. O., Some properties relating to the generalized Mittag-Leffler function, Adv. Appl. Math. Anal, 4(1) (2009), 21–30.

[25] Sharma, S. and Devi, M., Certain properties of extended Wright generalized hypergeometric function, Ann. Pure Appl. Math, 9 (2015), 45–51.

[26] Shukla, A. and Prajapati, J., On a generalization of Mittag-Leffler function and its properties, Journal of mathematical analysis and applications, 336(2) (2007), 797–811.

[27] Srivastava, H. M. and Choi, J., Zeta and q-Zeta functions and associated series and integrals, Elsevier, 2011.

[28] Srivastava, H. M. and Karlsson, P. W., Multiple gaussian hypergeometric series, 1985.

[29] Virchenko, N., Kalla, S. and Al-Zamel, A., Some results on a generalized hypergeometric function, Integral Transforms and Special Functions, 12(1) (2001), 89–100.